

On Protected Realizations of Quantum Information

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There are two complementary approaches to realizing quantum information so that it is protected from a given set of error operators. Both involve encoding information by means of subsystems. One is initialization-based error protection, which involves a quantum operation that is applied before error events occur [1]. The other is operator quantum error correction, which uses a recovery operation applied after the errors [2]. Together, the two approaches make it clear how quantum information can be stored at all stages of a process involving alternating error and quantum operations. In particular, there is always a subsystem that faithfully represents the desired quantum information. We give a definition of faithful realization of quantum information and show that it always involves subsystems. This justifies the “subsystems principle” for realizing quantum information [3]. In the presence of errors, one can make use of noiseless, (initialization) protectable, or error-correcting subsystems. We give an explicit algorithm for finding optimal noiseless subsystems by refining the strategy given in [4]. Finding optimal protectable or error-correcting subsystems is in general difficult. Verifying that a subsystem is error-correcting involves only linear algebra [2, 5, 6]. We discuss the verification problem for protectable subsystems and reduce it to a simpler version of the problem of finding error-detecting codes.

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I. INTRODUCTION

According to quantum information theory, quantum information is represented by states of a number of qudits, which are idealized d -level quantum systems. Here, we consider finite quantum information and do not explicitly refer to the underlying tensor-product structure of the state space of more than one qudit. Thus, we consider quantum information represented by the states of an ideal system I whose state space is determined by a Hilbert space \mathcal{H}_I of finite dimension N . In order to realize quantum information in a physical system P , two problems need to be solved. The first is to determine the ways in which the state space of P can usefully encode the desired quantum information, the second is to determine which among these ways can be best protected from decoherence due to the dynamics of P and its interactions. If quantum information is intended for use in quantum algorithms, a third problem is to ensure that the dynamics of the system and its interactions can be used to implement the desired quantum gates. Here we consider the first two problems. The mathematical conventions for objects and notations are explained at the end of the Introduction.

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A. Encoding Quantum Information and the Subsystems Principle

A general method for realizing or encoding quantum information is as a subsystem of \mathcal{H}_P [3]. This method involves a decomposition of \mathcal{H}_P as

$$\mathcal{H}_P = (\mathcal{H}_{I'} \otimes \mathcal{H}_S) \oplus \mathcal{H}_R, \quad (1)$$

where \oplus denotes the orthogonal sum of Hilbert spaces. This decomposition identifies I' as a subsystem of P , written as $I' \hookrightarrow P$. We use “primes” (as in I') for systems with identical state spaces. With this convention, Eq. (1) yields a representation of the states of the ideal quantum information system I in a subsystem of \mathcal{H}_P . We say that I is “encoded” in P and call the decomposition of Eq. (1) a “subsystem encoding”. For lack of a better word, we refer to S as the “cosubsystem” of I' in P . R is the “remainder system”. An example of such an encoding is that of a vibrational qubit of a single ion trapped in a one-dimensional harmonic potential. In this case the state space is spanned by the internal and vibrational levels of the ion. The space $\mathcal{H}_{I'} \otimes \mathcal{H}_S$ is formed from the first two vibrational levels and the internal levels of the ion, respectively. The space \mathcal{H}_R consists of states with more than two vibrational quanta irrespective of the internal state of the ion. The familiar cases of such encodings have the property that $\mathcal{H}_{I'}$ belongs to a physically meaningful degree of freedom. However, for the purpose of protecting against errors, it is usually necessary to use “entangled” encodings. An example is the noiseless qubit encoded in three spin-1/2 particles subject to collective decoherence [1]. A feature of subsystem encoding is that the states of I are not uniquely encoded as states of P . This is because any change of state of the cosubsystem S does not affect the states of I' .

Subsystem encodings of I in P are equivalent to \dagger -preserving isomorphic embeddings of $B(\mathcal{H}_I)$ into $B(\mathcal{H}_P)$ [1, 7]. Here $B(\mathcal{H})$ denotes the set of (bounded) operators on \mathcal{H} . In particular, given the subsystem encoding of Eq. (1), $B(\mathcal{H}_I)$ is isomorphic to the algebra of operators of the form $A \otimes \mathbf{I} \oplus \mathbf{0}$ with A acting on $\mathcal{H}_{I'}$, \mathbf{I} on \mathcal{H}_S and $\mathbf{0}$ on \mathcal{H}_R , where the operators are transported to \mathcal{H}_P via the isomorphism implicit in the subsystem decomposition as needed. Conversely, if \mathcal{A} is a subalgebra of bounded operators on \mathcal{H}_P and \mathcal{A} is \dagger -isomorphic to $B(\mathcal{H}_I)$, then there is a unique subsystem encoding such that the operators of \mathcal{A} are the operators of the form $A \otimes \mathbf{I} \oplus \mathbf{0}$ as above.

Are there ways of encoding quantum information that do not involve a subsystem encoding? In an attempt to answer this question, it is worth considering other prescriptions for encoding quantum information. There are two operationally defined ways of characterizing encoded information. The first is by a traditional encoding operation that isometrically embeds \mathcal{H}_I into \mathcal{H}_P . (Isometries are linear maps preserving the inner product.) This is the prescription used in the traditional theory of quantum error correction and corresponds to a subsystem encoding with trivial cosubsystem. In this case, the subspace $\mathcal{H}_{I'} \otimes \mathcal{H}_S = \mathcal{H}_{I'}$ of \mathcal{H}_P is known as a “quantum code”. That this is inadequate is apparent when one considers enlarging P by other relevant degrees of freedom. Subspaces also fail to capture the location of quantum information in realistic error-control settings, in particular fault-tolerant quantum computing. This is because in practice, error control never results in restoration of the encoded quantum information to any fixed quantum code. Assuming that this is a requirement leads to the conclusion that fault tolerance is not possible [8].

The second operational realization of quantum information involves specifying an ideal decoding procedure. Such a decoding procedure adjoins I and (possibly) an ancilla system A to P , where I and A are in specified initial states $|0\rangle_I$ and $|0\rangle_A$. The total state space is determined by $\mathcal{H}_P \otimes \mathcal{H}_I \otimes \mathcal{H}_A$. The decoding operation is a unitary operator on this state space. After it is applied, the desired quantum information resides in I . The decoding-based view of quantum information has been used successfully in analyses of fault-tolerant quantum architectures (see, for

example, [9]). To connect the decoding-based realization of quantum information to subsystems, note that the decoding operation is, in effect, an isometry from an extended space $\mathcal{H}_{P_e} \oplus \mathcal{H}_T$ to $\mathcal{H}_I \otimes \mathcal{H}_U$. Here we have identified \mathcal{H}_{P_e} with $\mathcal{H}_P \otimes |0\rangle_I \otimes |0\rangle_A$ and \mathcal{H}_U with $\mathcal{H}_A \otimes \mathcal{H}_P$. Decoding-based realization is therefore equivalent to subsystem encoding in an extension of the physical state space, where the extension need not be physically meaningful. One can consider generalizing decoding-based realizations by means of isometries that provide the identification

$$\mathcal{H}_P \otimes \mathcal{H}_I \oplus \mathcal{H}_T = \mathcal{H}_{I'} \otimes \mathcal{H}_S \oplus \mathcal{H}_R, \quad (2)$$

which is obtained if the decoding operation also involves additional physical systems in unspecified initial states. Although this is more general than subsystem encoding, most such isomorphisms do not result in quantum information that can be considered to be faithfully encoded in system P. We resolve this problem at the end of Section II by pairing decoding and encoding operations.

A third approach to encoding of quantum information uses operators to characterize quantum information and makes it possible to give a reasonable definition of “faithful encoding of I in P”. We give such a definition at the beginning of Sect. II and prove that every such encoding is associated with a subsystem. The intuition is that the states of I are characterized by the expectations of a linearly closed set of observables \mathcal{O} . To ensure the correct dynamics of these states, the complex multiples of the observables should form an algebra \dagger -isomorphic to $B(\mathcal{H})$. Thus, one expects that faithful encoding of quantum information requires identifying a \dagger -closed subalgebra \mathcal{A} of $B(\mathcal{H}_P)$ that is isomorphic to $B(\mathcal{H}_I)$. If this has been done, then the representation theory of \dagger -closed algebras uniquely identifies a decomposition $\mathcal{H}_P = \mathcal{H}_{I'} \otimes \mathcal{H}_S \oplus \mathcal{H}_R$ such that \mathcal{A} consists of all the operators acting only on $\mathcal{H}_{I'}$. The results of Sect. II provide support for the “subsystems principle” for realizing quantum information:

The subsystems principle: *Any faithful representation of quantum information in a physical system requires that at every point in time there are identifiable subsystems encoding the desired quantum information.*

The subsystem principle is powerful, but it is worth noting that it is sometimes convenient to use realizations of quantum information that do not satisfy this principle perfectly. For example, in optical quantum computing with “cat” states, it is convenient to represent the logical states of qubits by non-orthogonal coherent states [10, 11]. Another example is the study of “initialization-free” decoherence-free subsystems, where the probability amplitude of the encoded information may be less than 1 and the nature of the remaining amplitude must be taken into account [12].

B. Protecting Quantum Information

In most physical settings there are sources of errors that can affect encoded quantum information. Ideally, we would like exact knowledge of the error behavior of a physical system under all circumstances in which it is used. Since this knowledge is usually unavailable, one of a number of idealizing assumptions can be made. In the context of quantum channels, or when unwanted interactions are expected to have weak temporal correlations, we assume errors to be due to a known markovian process (in the continuous time setting) or a known quantum operation (in the discrete time setting). Both may be described by a collection of possible error events $\mathcal{E} = \{E_i\}$. In general, the goal of quantum error control is to find quantum information subsystems for which the effects of the markovian process or quantum operation can be suppressed to the largest extent possible. Because the exact nature of the errors is usually not known, this goal is typically difficult to pursue. To make the task more tractable, we can consider only those errors that are expected to

be likely and look for subsystems that allow for “good” protection against such errors. We can then bound the effect of other errors by making estimates of their maximum probability (or amplitude) of occurrence.

In this paper, we focus on subsystems that enable perfect protection against a fixed set of errors $\{E_i\}_i$, with or without active intervention. Because of the linearity of quantum mechanics, perfect protection against the E_i implies perfect protection against any error in the linear span \mathcal{E} of the E_i . Subsystems whose states are unaffected by the errors are known as “noiseless” or “decoherence free” subsystems [13] and were introduced in [1, 7] in the context of \dagger -closed \mathcal{E} (or the \dagger -closure of a non- \dagger -closed \mathcal{E}), in which case they can be characterized by irreducible representations of the commutant of \mathcal{E} , which is the set of operators that commute with all members of \mathcal{E} . In general, noiseless subsystems are not as easily characterized. In [14] an explicit characterization of noiseless subsystems for any \mathcal{E} is obtained. This characterization is readily seen to be equivalent to the statement that the subsystem l' of the decomposition $\mathcal{H}_P = \mathcal{H}_{l'} \otimes \mathcal{H}_S \oplus \mathcal{H}_R$ is noiseless if and only if the restriction of the E_i to the subspace $\mathcal{H}_{l'} \otimes \mathcal{H}_S$ acts as $\mathbf{I}^{(l')} \otimes E_i^{(S)}$. Several equivalent characterizations for when \mathcal{E} is the span of the operation elements of a specific quantum operation were obtained in [2, 5]. These characterizations do not directly address the question of how one can computationally search for noiseless subsystems. A strategy for this search was offered in [4]. This strategy requires finding \mathcal{E} -invariant subspaces and decomposing them into the canonical subsystems associated with the irreducible representations of a fixed-point algebra for a quantum operation whose operation elements span \mathcal{E} . In Sect. III we develop this strategy into an algorithm that does not require explicit constructions of algebras other than that generated by \mathcal{E} . The mathematical structure of algebras over the complex numbers plays a crucial role. Interestingly, if there exists a quantum operation whose operational elements span \mathcal{E} , then the algorithm simplifies substantially and is efficient in the dimension of the Hilbert space. Note that there is no a priori requirement that the likely errors included in \mathcal{E} be derived from a quantum operation. However, in most cases \mathcal{E} does satisfy this condition. To ensure that this condition holds, one can add $\mathbf{I} - \lambda \sum_i E_i^\dagger E_i$ for a sufficiently small λ , although the choice of spanning set E_i and λ may affect the availability of large-dimensional noiseless subsystems.

When no noiseless subsystem of sufficiently large dimension can be found, it is necessary to use active intervention to protect encoded quantum information. The idealized setting for active intervention involves alternating steps consisting of error events E_i and a quantum operation \mathcal{R} that ensures that the errors do not affect the encoded information. An operation \mathcal{R} with this property is known either as a “recovery” or as an “initialization” operation, depending on context. According to the subsystems principle, there must be two subsystems, one in which quantum information resides after error events but before \mathcal{R} is applied, and another after \mathcal{R} is applied. We call the first a “protectable” subsystem. The second is known as an “error-correcting” subsystem. Provided the encoded quantum information has been successfully protected, both subsystems are noiseless. The first is noiseless for the products $E_i R_j$, where the R_j are the operation elements of \mathcal{R} , whereas the second is noiseless for the operators $R_j E_i$. Protectable subsystems are defined (but not named) in [1], where it was shown how to determine the protectable subsystem in the case where the error-correcting subsystem is a quantum code, that is, the cosubsystem is one-dimensional. Error-correcting subsystems are the main feature of operator quantum error correction [2, 5] and directly generalize traditional error-correcting codes.

Knowledge of the protectable subsystem and the error-correcting subsystem associated with a recovery/initialization operation and the relationship between the two helps us to understand how quantum information is stored at all times. An advantage of the protectable subsystem is that in many cases it is a simple extension of the error-correcting subsystem. That is, the former’s

cosubsystem is a consistent extension of the latter's cosubsystem. As a result, the observables associated with the protectable subsystem induce the correct observables on the error-correcting subsystem. This implies that for the purpose of identifying the current value of the stored quantum information, it suffices to know the protectable subsystem, regardless of whether the last event was a recovery operation or an error. Examples of this situation are stabilizer codes with decoding algorithms based on syndrome extraction. It is readily verified that the associated protectable subsystem contains the stabilizer code as a subspace where the cosubsystem is in a particular state. In particular, this property holds for the stabilizer-based error-correcting subsystems identified in [15] and used to simplify Shor's 9-qubit one-error-correcting code [16], except that the error-correcting subsystem is defined by a subspace of the protectable subsystem's cosubsystem. In general, this relationship between protectable and error-correcting subsystems always holds if $\mathbf{I} \in \mathcal{E}$. It becomes particularly useful in the context of fault-tolerant quantum computation, where the recovery operation and error events can no longer be easily separated. In this case the ideal error-correcting codes or subsystems associated with a scheme are typically not where quantum information resides. It resides in the protectable subsystems of the scheme. Note that in this setting it is usually the case that the subsystems containing quantum information vary in time. This happens, for example, when teleportation is used for error correction, when quantum information is stored in memory versus being actively manipulated, and in cluster-state-based schemes as part of the model [17].

An advantage of error-correcting subsystems is that there are simple criteria and algorithms for determining whether there exists an associated recovery operation for which it becomes noiseless [2, 5, 6]. Not having to specify the recovery operation simplifies the search for subsystems suitable for protecting quantum information and makes it natural to talk about error-correcting subsystems without specifying the recovery operation. The same cannot be said for protectable subsystems. In Sect. IV we partially remedy this situation by reducing the problem of determining whether a subsystem is protectable to a number of other problems not involving the existence of a quantum operation.

C. Conventions

Capital letters in sans-serif font such as A, \dots, H, \dots, P are used to label quantum systems. The state space of a system A is determined by a Hilbert space, denoted by \mathcal{H}_A . We label states according to the quantum system they belong to. For example, $|\psi\rangle_A$ is a pure state of A and $\rho^{(A)}$ is a density matrix for A . The tensor product symbol \otimes may be omitted in tensor products of labeled states and operators. We frequently consider instances of identical state spaces realized by and in different systems. We use primes to distinguish the different systems with identical state spaces. Thus, I, I' and I'' are systems whose state spaces are identified via implicit isometries, which are inner-product-reserving linear maps. In particular, a state $|\psi\rangle_I$ of I is identified with the states $|\psi\rangle_{I'}$ and $|\psi\rangle_{I''}$ of systems I' and I'' . One way to interpret this is to consider ψ as a symbol labeling a vector in an appropriate Hilbert space \mathcal{H} and $|\psi\rangle \mapsto |\psi\rangle_I$ as the isometry identifying \mathcal{H} and \mathcal{H}_I . We use the equality symbol “=” not just to denote strict mathematical equality but also for identifying objects which are equal via an isomorphism. The isomorphisms involved are defined only implicitly, provided the meaning is clear. For a Hilbert space \mathcal{H} , $B(\mathcal{H})$ denotes the algebra of operators of \mathcal{H} . $U(\mathcal{H})$ denotes the group of unitary operators of \mathcal{H} . In this work, all state spaces are finite dimensional.

II. FAITHFUL ENCODINGS OF QUANTUM INFORMATION

To formalize the idea of “faithful encoding” we consider more general ways of encoding quantum information. A faithful encoding of \mathcal{I} in \mathcal{P} is a map D from density operators ρ on $\mathcal{H}_{\mathcal{I}}$ to non-empty sets of density operators on $\mathcal{H}_{\mathcal{P}}$ together with a map O from observables (hermitian operators) A of $\mathcal{H}_{\mathcal{I}}$ to non-empty sets of observables of an extension $\mathcal{H}_{\mathcal{Q}}$ of $\mathcal{H}_{\mathcal{P}}$ that satisfies three faithfulness requirements:

1. Statics: For all $\sigma \in D(\rho)$ and $X \in O(A)$,

$$\text{tr}(\sigma X) = \text{tr}(\rho A). \quad (3)$$

This requirement ensures that we can identify the expectation values of faithfully encoded states.

2. Unitary dynamics: For all $\sigma \in D(\rho)$ and $X \in O(A)$,

$$e^{-iX}\sigma e^{iX} \in D(e^{-iA}\rho e^{iA}). \quad (4)$$

With this requirement satisfied, we can evolve the states using conventional quantum control so that the evolved states are consistent with the first requirement.

For the next requirement, extend the domain of D to all positive semidefinite operators by defining $D(\mathbf{0}) = \{\mathbf{0}\}$ and for $\rho \neq \mathbf{0}$, $D(\rho) = \text{tr}(\rho)D(\rho/\text{tr}(\rho))$. For an operator Z , let $\Pi(Z, \lambda)$ be the projector onto the λ -eigenspace of Z , or, equivalently, the projector onto the null space of $Z - \lambda$. For λ not in the spectrum of Z , the projector is $\mathbf{0}$.

3. Measurement dynamics: For all $\sigma \in D(\rho)$ and $X \in O(A)$ and λ real,

$$\Pi(X, \lambda)\sigma\Pi(X, \lambda) \in D(\Pi(A, \lambda)\rho\Pi(A, \lambda)). \quad (5)$$

Faithful measurement dynamics ensures that projective measurements can be implemented correctly.

The support of a positive semidefinite Hermitian operator ρ is the span of its non-zero eigenvalue eigenvectors and is denoted by $\text{Supp}(\rho)$. For a set of such operators D , $\text{Supp}(D)$ is the span of the supports of the members of D .

Theorem 1 *If D and O are a faithful encoding of \mathcal{I} in \mathcal{P} , then one can identify a subsystem encoding $\mathcal{H}_{\mathcal{P}} = \mathcal{H}_{\mathcal{V}} \otimes \mathcal{H}_{\mathcal{S}} \oplus \mathcal{H}_{\mathcal{R}}$ such that for all ρ , $D(\rho)$ has support in $\mathcal{H}_{\mathcal{V}} \otimes \mathcal{H}_{\mathcal{S}}$, and for all A , $\mathcal{H}_{\mathcal{V}} \otimes \mathcal{H}_{\mathcal{S}}$ and $\mathcal{H}_{\mathcal{R}}$ are invariant subspaces of $O(A)$, and $O(A)$ acts as $A' \otimes \mathbf{I}$ on $\mathcal{H}_{\mathcal{V}} \otimes \mathcal{H}_{\mathcal{S}}$.*

The conclusion of the theorem does not hold if we assume only faithful statics and faithful unitary dynamics. For example, any irreducible representation of $U(\mathcal{H}_{\mathcal{I}})$ leads to an encoding satisfying these two faithfulness requirements, and such representations that have dimension larger than N exist. For example, if \mathcal{I} is a qubit, then any spin $> 1/2$ representation of $SU(2)$ yields an encoding that lacks faithful measurement dynamics. An other example is ensemble quantum computing with pure or pseudo-pure states [18, 19, 20]. In the case of pseudo-pure states, faithful statics is only satisfied up to a scale. Nevertheless, quantum information is still associated with subsystems. It may be interesting to determine the nature of encodings satisfying only faithful statics (perhaps weakened to allow for an overall scale factor) and faithful unitary dynamics. Are

they always equivalent to a sum of subsystems transforming under distinct irreducible representations of $U(\mathcal{H}_I)$? On the other hand, we conjecture that faithful statics and measurement dynamics imply faithful unitary dynamics. However, our proof of the theorem requires all three faithfulness properties.

An apparently more general faithfulness property, “faithful interactions”, requires that the encoding of I in P behaves correctly in interactions with other idealized systems. Faithful interactions are needed if we use the encoded quantum information in a quantum information processing setting with multiple physical systems, each encoding quantum information in some way. Faithful measurement dynamics can be seen to be a special case of faithful interactions, and, according to the theorem, it implies faithful interactions in general.

Proof of Thm. 1. Let \mathcal{V} be the linear sum of the supports of operators in $D(\rho)$ for all ρ . Let \mathcal{V}^\perp be its orthogonal complement. By assumption, $\mathcal{V} \subseteq \mathcal{H}_P$. The proof proceeds in three stages. In the first, we show that the operators of $O(A)$ have \mathcal{V} and \mathcal{V}^\perp as invariant subspaces. We can then redefine $O(A)$ by restricting its operators to \mathcal{V} . \mathcal{H}_R is identified as $\mathcal{V}^\perp \cap \mathcal{H}_P$. We then show that $O(A)$ consists of exactly one operator and deduce that O extends to an algebra isomorphism when restricted to commuting subsets of observables. The underlying reason for this involves showing that the eigenspaces of $O(A)$ may be faithfully identified with eigenspaces of A . The first two stages of the proof do not require faithful unitary dynamics. The last stage involves analyzing $SU(2)$ subgroups of $U(\mathcal{H}_I)$ and corresponding subgroups of $U(\mathcal{V})$ induced by O . Their action on eigenspaces of operators in the range of O implies the desired subsystem encoding.

For an operator X , let $\text{Null}(X)$ denote the null space of X .

Lemma 2 *Let $X \in O(A)$. Then $\text{Null}(X - \lambda) \cap \mathcal{V}$ is non-empty if and only if λ is in the spectrum of A . Furthermore, $\mathcal{V} = \sum_\lambda (\text{Null}(X - \lambda) \cap \mathcal{V})$, and $\text{Null}(X - \lambda) \cap \mathcal{V}$ is the linear span of the supports of $\rho \in D(\sigma)$ with $\text{Supp}(\sigma) \subseteq \text{Null}(A - \lambda)$.*

Proof. Suppose that λ is in the spectrum of A , and consider any $\rho \in D(\Pi(A, \lambda))$. By faithfulness of measurement dynamics, $\Pi(X, \lambda)\rho\Pi(X, \lambda) \in D(\Pi(A, \lambda))$. By faithfulness of statics, $\Pi(X, \lambda)\rho\Pi(X, \lambda)$ is not zero. Since the support of $\Pi(X, \lambda)\rho\Pi(X, \lambda)$ is contained in $\text{Null}(X - \lambda) \cap \mathcal{V}$, this intersection is non-zero. Conversely, suppose that $\text{Null}(X - \lambda) \cap \mathcal{V}$ is non-empty. Then there exist σ and $\rho \in D(\sigma)$ such that the support of ρ is not orthogonal to $\text{Null}(X - \lambda)$. Thus $\Pi(X, \lambda)\rho\Pi(X, \lambda)$ is not zero and is a member of $D(\Pi(A, \lambda)\sigma\Pi(A, \lambda))$. Because $D(\mathbf{0}) = \{\mathbf{0}\}$, $\Pi(A, \lambda)\sigma\Pi(A, \lambda)$ is not zero. Hence $\text{Null}(A - \lambda)$ is non-zero, so that λ is in the spectrum of A .

To prove that \mathcal{V} is spanned by the subspaces $\text{Null}(X - \lambda) \cap \mathcal{V}$, we use the following sequence of inclusions:

$$\begin{aligned}
 \mathcal{V} &\supseteq \sum_\lambda (\text{Null}(X - \lambda) \cap \mathcal{V}) \\
 &\supseteq \sum_\lambda \sum_\sigma \sum_{\rho: \rho \in D(\sigma)} (\text{Supp}(\Pi(X, \lambda)\rho\Pi(X, \lambda)) \cap \mathcal{V}) \\
 &= \sum_\sigma \sum_{\rho: \rho \in D(\sigma)} \sum_\lambda (\text{Supp}(\Pi(X, \lambda)\rho\Pi(X, \lambda)) \cap \mathcal{V}) \\
 &\supseteq \sum_\sigma \sum_{\rho: \rho \in D(\sigma)} \text{Supp}(\rho) \\
 &= \mathcal{V},
 \end{aligned} \tag{6}$$

(7)

where in each expression, λ ranges over the spectrum of A . The critical step in the sequence requires the inclusion

$$\sum_{\lambda} (\text{Supp}(\Pi(X, \lambda)\rho\Pi(X, \lambda)) \cap \mathcal{V}) \supseteq \text{Supp}(\rho). \quad (8)$$

To prove this inclusion, observe that $\text{Supp}(\Pi(X, \lambda)\rho\Pi(X, \lambda)) \subseteq \mathcal{V}$ because $\Pi(X, \lambda)\rho\Pi(X, \lambda) \in D(\Pi(A, \lambda)\rho\Pi(A, \lambda))$. Faithfulness of measurement dynamics implies that $\Pi(X, \lambda)\rho\Pi(X, \lambda) = \mathbf{0}$ for λ not in the spectrum of A . It then suffices to recall that for a complete set of orthogonal projectors P_i , $\sum_i \text{Supp}(P_i\rho P_i) = \text{Supp}(\sum_i P_i\rho P_i) \supseteq \text{Supp}(\rho)$.

For the last claim of the lemma, let W be the set of density operators with support in $\text{Null}(A - \lambda)$. If $\sigma \in W$, then $\Pi(A, \lambda)\sigma\Pi(A, \lambda) = \sigma$ and for all $\lambda' \neq \lambda$, $\Pi(A, \lambda')\sigma\Pi(A, \lambda') = \mathbf{0}$. Faithful measurement dynamics imply that for $\rho \in D(\sigma)$, $\text{Supp}(\rho) \subseteq \text{Null}(X - \lambda)$. Thus, $\sum_{\sigma \in W} \sum_{\rho \in D(\sigma)} \text{Supp}(\rho) \subseteq \text{Null}(X - \lambda) \cap \mathcal{V}$. The following sequence of relationships proves the reverse inclusion:

$$\begin{aligned} \text{Null}(X - \lambda) \cap \mathcal{V} &= \text{Null}(X - \lambda) \cap \left(\sum_{\sigma} \sum_{\rho: \rho \in D(\sigma)} \text{Supp}(\rho) \right) \\ &\subseteq \Pi(X, \lambda) \left(\sum_{\sigma} \sum_{\rho: \rho \in D(\sigma)} \text{Supp}(\rho) \right) \\ &= \sum_{\sigma} \sum_{\rho: \rho \in D(\sigma)} \text{Supp}(\Pi(X, \lambda)\rho\Pi(X, \lambda)) \\ &\subseteq \sum_{\sigma} \sum_{\rho: \rho \in D(\Pi(A, \lambda)\sigma\Pi(A, \lambda))} \text{Supp}(\rho) \\ &= \sum_{\sigma \in W} \sum_{\rho: \rho \in D(\sigma)} \text{Supp}(\rho), \end{aligned} \quad (9)$$

where we have used the fact that for a projector Π and a positive semidefinite hermitian operator ρ , $\Pi\text{Supp}(\rho) = \text{Supp}(\Pi\rho\Pi)$. ■

Corollary 3 *Let $X \in O(A)$. Then \mathcal{V} and \mathcal{V}^\perp are invariant subspaces of X .*

Proof. Because the eigenspaces of X are orthogonal, Lemma 2 implies that X can be block diagonalized with respect to an orthonormal basis whose first members span \mathcal{V} . ■

Lemma 2 and Cor. 3 imply that without loss of generality, we can assume that for all $X \in O(A)$, X restricted to \mathcal{V}^\perp is $\mathbf{0}$. If not, replace every member of $O(A)$ with its restriction to \mathcal{V} . This does not affect any of the faithfulness requirements.

The last statement in Lemma 2 together with the assumption that $X \in O(A)$ has trivial action on \mathcal{V}^\perp implies that X 's eigenspaces and eigenvalues are determined by A and the map D . It follows that $O(A)$ consists of exactly one operator. Thus, without loss of generality, we now take $O(A)$ to be a function from observables of \mathcal{H}_I to observables of \mathcal{H}_P . Lemma 2 also implies that inclusion relationships between eigenspaces of observables of \mathcal{H}_I are preserved by O .

Corollary 4 *Suppose that $\text{Null}(A - \lambda_1) \subseteq \text{Null}(B - \lambda_2)$. Then $\text{Null}(O(A) - \lambda_1) \subseteq \text{Null}(O(B) - \lambda_2)$.*

If observables A and B commute, we can construct an observable C whose eigenspaces are the maximal common eigenspaces of A and B . By using the eigenspaces of $O(C)$ to derive the eigenspaces of $O(A)$ and of $O(B)$, we can see that $O(AB) = O(A)O(B)$ and $O(\alpha A + \beta B) = O(\alpha A) + O(\beta B)$, so that O preserves the algebraic structure of commuting sets of observables. Similarly, for any eigenbasis $|\lambda_i\rangle$ of A , we can use an operator C with non-degenerate eigenvalues having the same eigenbasis to see that the spaces $\text{Supp}(D(|\lambda_i\rangle\langle\lambda_i|))$ are a complete orthogonal decomposition of \mathcal{V} into eigenspaces of A . From this it follows that $O(A)$ is determined by the values of D on pure states.

For the last stage of the proof of Thm. 1, we fix an orthonormal basis $|i\rangle_l$ of \mathcal{H}_l . Let $e_{ij} = |i\rangle_l\langle j|$, $X_{ij} = e_{ij} + e_{ji}$, $Y_{ij} = -ie_{ij} + ie_{ji}$, $C = \sum_i ie_{ii}$ and $\mathcal{V}_i = \text{Null}(O(C) - i)$. Note that $ie^{-iX_{ij}\pi/2}|i\rangle_l = e^{-iY_{ij}\pi/2}|i\rangle_l = |j\rangle_l$. According to faithfulness of unitary dynamics, $e^{-iO(A)}D(\sigma)e^{iO(A)} \subseteq D(e^{-iA}\sigma e^{iA})$. (For sets D and operators U , $UD = \{Ux : x \in D\}$.) The inclusion is an equality because we also have $D(e^{-iA}\sigma e^{iA}) = e^{-iO(A)}e^{iO(A)}D(e^{-iA}\sigma e^{iA})e^{-iO(A)}e^{iO(A)} \subseteq e^{-iO(A)}D(\sigma)e^{iO(A)}$. This and the earlier results imply that $e^{-iO(X_{ij})\pi/2}\mathcal{V}_i = \mathcal{V}_j$. By using Cor. 4 with the eigenspaces of X_{ij} , Y_{ij} and $e_{ii} + e_{jj}$, we can see that the non-zero eigenspaces of $O(X_{ij})$ and $O(Y_{ij})$ are contained in $\mathcal{V}_i \oplus \mathcal{V}_j$.

Because of the algebraic properties of O mentioned above and $X_{ij}^2 = e_{ii} + e_{jj}$, $e^{-iO(X_{ij})\pi/2} = -iO(X_{ij})$. We can therefore fix an orthonormal basis $|il\rangle$ of \mathcal{V}_i such that $O(X_{0i})|0l\rangle = |il\rangle$ and $O(X_{0i})|il\rangle = |0l\rangle$. The goal is to show that we can identify $|il\rangle$ with $|i\rangle_{l'}|l\rangle_S$ such that $O(A)$ acts as the identity on the cosubsystem S . Note that the operators X_{0j} and e_{jj} generate the Lie algebra of $U(\mathcal{H}_l)$. Thus compositions of exponentials of the form $e^{-iX_{0j}t}$ or $e^{-ie_{jj}s}$ act transitively on the pure states of \mathcal{H}_l . It follows that for any $|\psi\rangle_l$, $\text{Supp}(D(|\psi\rangle_l\langle\psi|))$ is an image of corresponding compositions of exponentials of the form $e^{-iO(X_{0i})t}$ or $e^{-ie_{jj}s}$ acting on \mathcal{V}_0 . Such compositions are completely determined by the basis $|il\rangle$. Now $O(A)$ is determined by $\text{Supp}(D(|\psi\rangle_l\langle\psi|))$, with $|\psi\rangle_l$ ranging over eigenvectors of $O(A)$. Since for fixed l , $O(X_{0i})$ and $O(e_{jj})$ act as they should on the states $|kl\rangle$, we have that $O(A)$ necessarily satisfies $\langle kl|O(A)|kl\rangle = \langle k|A|k\rangle$, as desired. ■

We return to the issue of the relationship between decoding operations and faithful encodings. Decoding as defined in the introduction is the traditional way of identifying quantum information and generalizes recovery operations. A general form of the situation addressed by decoding involves one or more encoding isometries $C_i : \mathcal{H}_l \rightarrow \mathcal{H}_P$, one or more possible events E_i that are operators on \mathcal{H}_P that may occur before we decode, and a decoding operation D that (after purification, if necessary) isometrically maps \mathcal{H}_P into $\mathcal{H}_l \otimes \mathcal{H}_A$ for some possibly composite system A . We say that $\{C_i\}$, $\{E_j\}$, D preserve quantum information if for all i, j , $DE_jC_i|\psi\rangle_l = |\psi\rangle_l|\phi_{ij}\rangle_A$ for some unnormalized vector $|\phi_{ij}\rangle_A$ that does not depend on $|\psi\rangle_l$. Here, which event E_i occurred is assumed to be unknown. We could consider the case where the decoding operation is chosen after the events and depends on partial knowledge of the events. However, by conditioning on the knowledge, we return to the situation just described. The only difference is that the subsystem associated with the situation may depend on the partial knowledge. To capture the case where quantum information is stored in error-correcting subsystems, let C_i be given by the isometries identifying \mathcal{H}_l with $\mathcal{H}_{l'} \otimes |i\rangle_S$, where the $|i\rangle_S$ range over any spanning set of \mathcal{H}_S .

In order to justify the subsystems principle, we prove the next theorem.

Theorem 5 *If $\{C_i\}$, $\{E_j\}$, D preserve quantum information, then there exists a subsystem encoding $\mathcal{H}_P = \mathcal{H}_{l'} \otimes \mathcal{H}_{S'} \oplus \mathcal{H}_R$ such that for all i, j and $|\psi\rangle_l$, $E_jC_i|\psi\rangle_l = |\psi\rangle_{l'}|\phi'_{ij}\rangle_{S'}$, $\in \mathcal{H}_P$ and $D|\psi\rangle_{l'}|\phi'_{ij}\rangle_{S'} = |\psi\rangle_l|\phi_{ij}\rangle_A$, where the $|\phi'_{ij}\rangle_{S'}$ and $|\phi_{ij}\rangle_A$ do not depend on $|\psi\rangle_l$.*

Proof. This follows from the fact that there exists a protectable subsystem associated with any quantum error-correcting code and associated recovery operation, which was proven in [1, 21]. Alternatively, we could prove the theorem from Thm. 1 by defining $D(|\psi\rangle) = \{E_j C_i |\psi\rangle\}$ and $O(A)$ by pullback of the appropriate operators via the decoding operator D . Here we give a direct proof. Let the $|\phi_{ij}\rangle_A$ be as required according to the definition of preserving quantum information. Let \mathcal{S} be the set of vectors $|\phi\rangle_A$ such that $\mathcal{H}_I \otimes |\phi\rangle_A$ is contained in the range of D . Then \mathcal{S} contains the $|\phi_{ij}\rangle_A$. Because the range of D is linearly closed, so is \mathcal{S} . Define $\mathcal{H}_S = \mathcal{S}$. Using the isometric properties of D , we can define a subsystem encoding $\mathcal{H}_P = \mathcal{H}_{I'} \otimes \mathcal{H}_{S'} \oplus \mathcal{H}_R$ such that $D(|\psi\rangle_{I'}, |\phi\rangle_{S'}) = |\psi\rangle_I |\phi\rangle_S$. This subsystem encoding has the desired properties. ■

III. FINDING NOISELESS SUBSYSTEMS

If \mathcal{A} is a \dagger -closed subalgebra of $B(\mathcal{H}_P)$, the canonical decomposition of \mathcal{H}_P is

$$\mathcal{H}_P = \sum_i \mathcal{H}_{I_i} \otimes \mathcal{H}_{S_i} \oplus \mathcal{H}_R, \quad (10)$$

where operators $A \in \mathcal{A}$ act as $\sum_i \mathbf{I}^{(I_i)} \otimes S_i(A)^{(S_i)} + \mathbf{0}^{(R)}$. For every operator of the form $\sum_i \mathbf{I}^{(I_i)} \otimes B_i^{(S_i)} + \mathbf{0}^{(R)}$, there exists an $A \in \mathcal{A}$ with $S_i(A) = B_i$. The \mathcal{H}_{I_i} are noiseless subsystems for \mathcal{A} . We also consider \mathcal{H}_R to be noiseless for \mathcal{A} , but note that error operators in \mathcal{A} have probability zero for states in this subspace. The tensor products and direct sums in the decomposition must be consistent with the Hilbert space's inner product. This is implicit in the construction and the identification via an isometry.

Let \mathcal{E} be a linearly closed set of error operators in $B(\mathcal{H}_P)$. For now, we do not assume that \mathcal{E} is the span of the operation elements of a quantum operation. Let $\mathcal{H}_{I'} \otimes \mathcal{H}_S \oplus \mathcal{H}_R$ define a subsystem encoding of I in P . Let Π be the projector onto the support of I' , $\mathcal{H}_{I'} \otimes \mathcal{H}_S \subseteq \mathcal{H}_P$. The subsystem is noiseless for \mathcal{E} if and only if for all $E \in \mathcal{E}$, the restriction of E to $\mathcal{H}_{I'} \otimes \mathcal{H}_S$ acts as the identity on $\mathcal{H}_{I'}$. Equivalently, for all $E \in \mathcal{E}$, $E\Pi = \mathbf{I}^{(I)} \otimes S(E)^{(S)}$. It is straightforward to verify that if the subsystem is noiseless, then Π projects onto an invariant subspace of \mathcal{E} and $\mathcal{E}\Pi$ generates a \dagger -closed subalgebra of operators acting on the support of Π whose canonical decomposition contains noiseless subsystems with state space dimension at least N , the dimension of $\mathcal{H}_{I'}$. Such noiseless subsystems are also noiseless for \mathcal{E} . This leads to a strategy for finding noiseless subsystems with maximum dimensional $\mathcal{H}_{I'}$ that is equivalent to the strategy proposed in [4]: 1. Pick an invariant subspace of \mathcal{E} and let Π be its projector. 2. Determine the canonical decomposition of the \dagger -closed algebra generated by $\Pi\mathcal{E}$. The noiseless subsystems of this algebra are candidate noiseless subsystems for \mathcal{E} . Our goal is to provide an explicit algorithm for finding suitable Π and associated subsystems. The algorithm involves the decomposition of a matrix algebra, for which efficient algorithms are known, as we explain below. Note that in addition to the noiseless subsystems identified in this way, one can construct other noiseless subsystems as subsystems of already obtained noiseless subsystems, or by combining cosubsystems of identical dimensional noiseless subsystems with orthogonal supports. These constructions cannot yield larger dimensional noiseless subsystems, but they may generate ones with greater error tolerance or more efficiently controllable states.

Let \mathcal{A} be the algebra generated by \mathcal{E} . Any noiseless subsystem for \mathcal{E} is a noiseless subsystem for \mathcal{A} . \mathcal{A} is not necessarily \dagger -closed. As a result, \mathcal{A} does not have a canonical decomposition

of \mathcal{H}_P as a direct sum of tensor products of Hilbert spaces. Nevertheless, we can identify a special subspace within which a similar decomposition is possible and where maximum dimensional noiseless subsystems may be found. This subspace is the span \mathcal{S} of the irreducible subspaces of \mathcal{A} . A subspace \mathcal{V} of \mathcal{H}_P is “irreducible” for \mathcal{A} if it is \mathcal{A} -invariant, $\mathcal{A}\mathcal{V} \neq 0$ and there is no non-zero \mathcal{A} -invariant proper subspace of \mathcal{V} . Let \mathcal{Z} be the null space of \mathcal{A} . Both \mathcal{S} and \mathcal{Z} are invariant.

Lemma 6 *A maximum dimensional noiseless subsystem for \mathcal{A} can be found in \mathcal{S} or in \mathcal{Z} .*

Note that \mathcal{Z} is itself a noiseless subsystem. This subsystem is trivial in the sense that the probability of \mathcal{E} -errors is zero for any state in \mathcal{Z} . This means that in a realistic setting, there must be operators acting on the system not included in \mathcal{E} , and for \mathcal{Z} to be at least approximately noiseless, they need to act as operators close to the identity when restricted to \mathcal{Z} .

Proof of Lemma 6. Suppose that \mathcal{H}_V is a noiseless subsystem of \mathcal{H}_P with cosubsystem \mathcal{H}_S . Then $\mathcal{V} = \mathcal{H}_V \otimes \mathcal{H}_S$ is invariant under \mathcal{A} and for $A \in \mathcal{A}$, A acts as $\mathbf{I}^{(V)} \otimes S(A)^{(S)}$ on \mathcal{V} . If for all $A \in \mathcal{A}$, $S(A)^{(S)} = 0$, then $\mathcal{V} \subseteq \mathcal{Z}$ and we are done. If not, then there exists a nontrivial irreducible subspace \mathcal{S}_i of \mathcal{H}_S under the action of $\{S(A): A \in \mathcal{A}\}$. For each state $|\psi\rangle_V$, $|\psi\rangle_V \otimes \mathcal{S}_i$ is an irreducible representation for \mathcal{A} . In particular, $\mathcal{H}_V \otimes \mathcal{S}_i \subseteq \mathcal{S}$. Since $\mathcal{H}_V \otimes \mathcal{S}_i$ is also a noiseless subsystem, the proof is complete. ■

According to the theory of R -modules, \mathcal{S} is a module for \mathcal{A} . The definition implies that it is semisimple, from which it follows that $\mathcal{S} = \sum_i \mathcal{S}_i$ where the \mathcal{S}_i are irreducible and the sum is over independent subspaces, see [22], Chapter 9. \mathcal{S}_i and \mathcal{S}_j are isomorphic with respect to the action of \mathcal{A} if there exists an invertible linear map U_{ij} from \mathcal{S}_i to \mathcal{S}_j such that for $|x\rangle \in \mathcal{S}_i$, $AU_{ij}|x\rangle = U_{ij}A|x\rangle$. The map U_{ij} is said to “intertwine” \mathcal{S}_i and \mathcal{S}_j . We can relabel the \mathcal{S}_i to form sets $\{\mathcal{S}_{ik}\}_i$ of isomorphic irreducible representations. For each k , let \mathcal{V}_k be the span of the \mathcal{S}_{ik} and let $U_{0j}^{(k)}$ be an intertwiner from \mathcal{S}_{0k} to \mathcal{S}_{jk} . Choose a basis $|i0k\rangle$ of \mathcal{S}_{0k} and define $|ijk\rangle = U_{0j}^{(k)}|i0k\rangle$. Note that these vectors need not be orthogonal or normalized. Nevertheless, they define invertible linear maps from tensor products $\mathcal{J}_k \otimes \mathcal{S}_{0k}$ to \mathcal{V}_k via the linear extension of $|j\rangle \otimes |i0k\rangle \mapsto |ijk\rangle$. The action of $A \in \mathcal{A}$ with respect to this factorization is on \mathcal{S}_{0k} only.

Lemma 7 *A maximum dimensional noiseless subsystem for \mathcal{A} in \mathcal{S} can be found in one of the \mathcal{V}_k .*

Proof. This follows from the argument given in the proof of Lemma 6. It suffices to observe that the irreducible representations $|\psi\rangle_{\mathcal{J}} \otimes \mathcal{S}_i$ are isomorphic for different $|\psi\rangle_{\mathcal{J}}$. ■

The main remaining problem in narrowing the search space for maximum dimensional noiseless subsystems is that the factorization of the \mathcal{V}_k may fail to preserve the inner product. To simplify the notation, fix k and let $\mathcal{V} = \mathcal{V}_k$, $\mathcal{S}_0 = \mathcal{S}_{0k}$ and $\mathcal{J} = \mathcal{J}_k$. Let U be an invertible linear map from $\mathcal{J} \otimes \mathcal{S}_0$ to \mathcal{V} that implements the above-mentioned factorization of \mathcal{V} . Thus, for $A \in \mathcal{A}$ and $|x\rangle \in \mathcal{J} \otimes \mathcal{S}_0$, $AU|x\rangle = U(\mathbf{I} \otimes R(A))|x\rangle$, where R is a well-defined, irreducible representation of \mathcal{A} on \mathcal{S}_0 . Note that an irreducible representation of \mathcal{A} on \mathcal{S}_0 is onto $B(\mathcal{S}_0)$ (Burnside’s theorem). This implies that any noiseless subsystem of \mathcal{V} must be associated with a subspace \mathcal{J}' of \mathcal{J} such that the restriction of U to $\mathcal{J}' \otimes \mathcal{S}_0$ has the property that there are linear operators W on \mathcal{J}' and V on \mathcal{S}_0 such that $U(W \otimes V)$ is an isometry. Fortunately, in cases where \mathcal{A} is generated by the operational elements of a quantum operation, we do not need to search for such subspaces.

Lemma 8 *If \mathcal{A} is generated by the operational elements of a quantum operation, then there exist linear operators W on \mathcal{J} and V on \mathcal{S}_0 such that $U(W \otimes V)$ is unitary.*

Proof. Let $\{E_i\}_i$ generate \mathcal{A} , where the E_i are the operational elements of a quantum operation \mathcal{O} . By composing \mathcal{O} with itself sufficiently many times, it is possible to obtain a quantum operation \mathcal{O}' such that its operational elements span \mathcal{A} . Thus, without loss of generality, assume that the E_i span \mathcal{A} and $\sum_i E_i^\dagger E_i = \mathbf{I}$. We have $E_i = U(\mathbf{I} \otimes R(E_i))U^{-1}$. In order to continue, assume, without loss of generality, that $\mathcal{J} \otimes \mathcal{S}_0 = \mathcal{V}$. This can be done by means of any isometry between \mathcal{V} and $\mathcal{J} \otimes \mathcal{S}_0$. This implies that U is an invertible but not necessarily unitary linear map from $\mathcal{J} \otimes \mathcal{S}_0$ to itself. We have

$$\sum_i U^{-1\dagger}(\mathbf{I} \otimes R(E_i)^\dagger)U^\dagger U(\mathbf{I} \otimes R(E_i))U^{-1} = \mathbf{I}, \quad (11)$$

or, equivalently,

$$\sum_i (\mathbf{I} \otimes R(E_i)^\dagger)U^\dagger U(\mathbf{I} \otimes R(E_i)) = U^\dagger U. \quad (12)$$

This implies that for all positive semidefinite σ on \mathcal{J} ,

$$\sum_i R(E_i)^\dagger \text{tr}_{\mathcal{J}}((\sigma \otimes \mathbf{I})U^\dagger U(\sigma \otimes \mathbf{I}))R(E_i) = \text{tr}_{\mathcal{J}}((\sigma \otimes \mathbf{I})U^\dagger U(\sigma \otimes \mathbf{I})), \quad (13)$$

where $\text{tr}_{\mathcal{J}}$ is the partial trace over \mathcal{J} . Let \mathbf{R} be the operation defined by $\mathbf{R}(X) = \sum_i R(E_i)^\dagger X R(E_i)$. According to Eq. 13, for all positive semidefinite σ , $\text{tr}_{\mathcal{J}}((\sigma \otimes \mathbf{I})U^\dagger U(\sigma \otimes \mathbf{I}))$ is a positive semidefinite fixed point of \mathbf{R} . The spanning assumption on the E_i and irreducibility of \mathcal{S}_0 under $R(\mathcal{A})$ imply that the $R(E_i)$ span $B(\mathcal{S}_0)$. It follows that if $\rho \neq \mathbf{0}$ is positive semidefinite and $\mathbf{R}(\rho) = \rho$, then the support of ρ is \mathcal{S}_0 . It also implies that \mathbf{R} has at most one positive fixed point (up to positive multiples): If ρ' is another one, then so is $\rho - \epsilon\rho'$ for all ϵ . Let ϵ be the largest such that $\rho - \epsilon\rho'$ is positive semidefinite. Then $\rho - \epsilon\rho'$ is a fixed point with non-maximal support, which implies that it is $\mathbf{0}$. Let ρ be the unique trace 1 positive fixed point of \mathbf{R} . Then, for all positive semidefinite σ , $\text{tr}_{\mathcal{J}}((\sigma \otimes \mathbf{I})U^\dagger U(\sigma \otimes \mathbf{I}))$ is a multiple of ρ . We can now deduce that $U^\dagger U = \rho' \otimes \rho$ for some strictly positive ρ' . Defining $V = \rho^{-1/2}$ and $W = \rho'^{-1/2}$ yields the lemma. ■

The above suggests the following strategy for finding maximum-dimensional noiseless subsystems: 1. Determine the span \mathcal{S} of the irreducible subspaces of \mathcal{A} . 2. Decompose \mathcal{S} into a direct sum $\bigoplus_i \mathcal{I}_i$ of subspaces spanned by isomorphic irreducible subspaces. 3. For each \mathcal{I}_i , let \mathcal{A}_i be the restriction of \mathcal{A} to \mathcal{I}_i and find the canonical decomposition for the \dagger -closed algebra generated by \mathcal{A}_i . This strategy will find maximum-dimensional noiseless subsystems provided that \mathcal{A} is generated by the operational elements of a quantum operation. There are efficient algorithms for each step of this strategy; for a review, see [23]. For completeness, we outline an algorithm that implements the strategy.

To find \mathcal{S} , consider the structure of \mathcal{A} in more detail. If \mathcal{A} does not contain \mathbf{I} , replace \mathcal{A} by $\mathcal{A} + \mathbb{C}\mathbf{I}$. By doing so, the action of \mathcal{A} on \mathcal{Z} is no longer zero, but \mathcal{Z} is still distinguishable from the other irreducible subspaces. Every one-dimensional subspace of \mathcal{Z} is irreducible and not isomorphic to the irreducible subspaces of \mathcal{S} . There exists a maximal chain of invariant subspaces $0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \dots \subset \mathcal{V}_n = \mathcal{H}_{\mathcal{P}}$ such that the action of \mathcal{A} induced on the quotients $\mathcal{V}_{k+1}/\mathcal{V}_k$ is irreducible or zero. In a basis $|e_{kj}\rangle$ of $\mathcal{H}_{\mathcal{P}}$ where $|e_{(k+1)j}\rangle \in \mathcal{V}_{k+1} \setminus \mathcal{V}_k$ (\setminus denotes set difference), the operators of \mathcal{A} are block upper triangular. Let \mathcal{J} be the members of \mathcal{A} that act as $\mathbf{0}$ on each of these quotients. \mathcal{J} is known as the Jacobson radical of \mathcal{A} . Let \mathcal{N} be the null space of \mathcal{J} , which is the set of vectors in the intersection of the null spaces of operators of \mathcal{J} . Then \mathcal{N} is invariant (because \mathcal{J} is a two-sided ideal) and $\mathcal{S} \subseteq \mathcal{N}$ (because \mathcal{S} is invariant and the span of irreducible subspaces).

A fundamental property of \mathcal{J} is that \mathcal{A}/\mathcal{J} is a semisimple algebra. Let $\mathcal{A}_{\mathcal{N}}$ be the restriction of \mathcal{A} to \mathcal{N} . Then $\mathcal{A}_{\mathcal{N}}$ is isomorphic to a quotient of \mathcal{A}/\mathcal{J} , which implies that $\mathcal{A}_{\mathcal{N}}$ is semisimple. According to the representation theory of semisimple algebras, \mathcal{N} is a semisimple $\mathcal{A}_{\mathcal{N}}$ -module, which implies that $\mathcal{N} = \mathcal{S}$. Thus, to determine \mathcal{S} , we can use an efficient algorithm for finding the Jacobson radical and then compute its null space. Decomposing \mathcal{N} into independent irreducible subspaces can be done by means of an efficient algorithm for the decomposition of semisimple algebras over the complex numbers. A randomized algorithm can be based on the observation that $\mathcal{A}_{\mathcal{N}}$ is isomorphic to a direct sum of complete matrix algebras \mathcal{A}_k on the \mathcal{S}_{0k} , acting canonically on the (non-unitary) decomposition of \mathcal{S} into a direct sum of products $\mathcal{I}_k = \mathcal{J}_k \otimes \mathcal{S}_{0k}$. It follows that a random matrix in $\mathcal{A}_{\mathcal{N}}$ (with respect to a suitably chosen probability distribution) typically has generalized eigenspaces that generate (by multiplication by members of $\mathcal{A}_{\mathcal{N}}$) exactly one of the invariant subspaces $\mathcal{I}_k = \mathcal{J}_k \otimes \mathcal{S}_{0k}$. This yields the desired matrix algebras \mathcal{A}_k . For each \mathcal{A}_k , let \mathcal{A}_k^* be the \dagger -closed algebra generated by \mathcal{A}_k . The canonical factorization of \mathcal{I}_k with respect to \mathcal{A}_k^* can also be obtained by a randomized algorithm. By construction, $\mathcal{I}_k = \mathcal{H}_1 \otimes \mathcal{H}_2$ (isometrically), with \mathcal{A}_k^* acting only on \mathcal{H}_2 . The eigenspaces of a randomly chosen Hermitian operator H_2 in \mathcal{A}_k^* are typically of the form $\mathcal{H}_{1i} \otimes |i\rangle$ for an orthonormal basis of \mathcal{H}_2 , where $\mathcal{H}_{1i} = \mathcal{H}_1$, but with the isometry for making this identification not yet known. With high probability, these isometries can be determined from an independently chosen second H'_2 by expressing H'_2 in an orthonormal basis whose i 'th block of vectors is a basis of $\mathcal{H}_{1i} \otimes |i\rangle$. Because H'_2 is a Kronecker product with identity action on \mathcal{H}_1 , the i, j block of H'_2 must define an isometry between \mathcal{H}_{1i} and \mathcal{H}_{1j} (if it is nonzero). These isometries must be consistent and induce the desired tensor product structure.

Components of the algorithm of the previous paragraph not given explicitly include the generation of an algebra from a set of matrices (this comes up in generating \mathcal{A} from an error set and generating \dagger -closed algebras from a given one) and various standard matrix manipulations such as matrix multiplication, eigenvalue and eigenspace determination, etc. We do not discuss the latter here. To generate the matrix algebra from a set of operators $\{E_i\}$, assume without loss of generality that the E_i are independent. Then iteratively, choose i, j and determine whether $E_i E_j$ is in the span of the E_i . If not, adjoin it to the set. Stop when for all i, j , $E_i E_j$ is in the span of the E_i .

IV. PROTECTABLE SUBSYSTEMS

As above, let $\{E_i\}_i$ be a set of error operators on $\mathcal{H}_{\mathcal{P}}$. Let $\mathcal{H}_{\mathcal{P}} = \mathcal{H}_{\mathcal{V}} \otimes \mathcal{H}_{\mathcal{S}} \oplus \mathcal{H}_{\mathcal{R}}$ be a subsystem encoding. The subsystem \mathcal{V} is “initialization protectable” (or “protectable” for short), if there exists a quantum operation with operation elements $\{R_i\}_i$ such that \mathcal{V} is noiseless for $\{E_i R_j\}_{i,j}$. The goal of this section is to reduce the problem of determining whether a given subsystem is protectable to the problem of searching for certain extremal error-detecting codes. We then reduce this problem to several linear algebra problems.

Let $|i\rangle_{\mathcal{S}}$ be an orthonormal basis of $\mathcal{H}_{\mathcal{S}}$. For any state $|\psi\rangle$ of $\mathcal{H}_{\mathcal{V}} \otimes \mathcal{H}_{\mathcal{S}} \subseteq \mathcal{H}_{\mathcal{P}}$, we define $\mathcal{S}\langle i|\psi\rangle \in \mathcal{H}_{\mathcal{V}}$ by the identity $\sum_i (\mathcal{S}\langle i|\psi\rangle) \otimes |i\rangle_{\mathcal{S}} = |\psi\rangle$. Let \mathcal{V} be the intersection of the inverse images of $\mathcal{H}_{\mathcal{V}} \otimes \mathcal{H}_{\mathcal{S}}$ under the errors E_i .

Lemma 9 *With the definitions of the previous paragraphs, \mathcal{V} is protectable if and only if there exists a subspace $\mathcal{D} \subseteq \mathcal{V}$ with the property that the maps $F_{ij} : |\psi\rangle \mapsto \mathcal{S}\langle j|E_i|\psi\rangle$ are proportional to a single isometry from \mathcal{D} to $\mathcal{H}_{\mathcal{V}}$.*

Proof. For the “if” part of the lemma, we show that \mathcal{D} is an error-correcting code for $\{E_i\}_i$. We can reconstruct E_i on \mathcal{D} from the F_{ij} by the identity $E_i|\psi\rangle = \sum_j (F_{ij}|\psi\rangle) \otimes |j\rangle_{\mathcal{S}}$. Let U be

the isometry such that ${}^S\langle j|E_i|\psi\rangle = \alpha_{ij}U|\psi\rangle$. Then $E_i|\psi\rangle = (U|\psi\rangle) \otimes \sum_j \alpha_{ij}|j\rangle_S$. That \mathcal{D} is an error-correcting code follows immediately. The operators R_i are given by $U^{-1}|i\rangle_S\langle i|$.

For the converse, we can use the subsystems principle (more specifically, Thm. 5), according to which there must be a subsystem decomposition $\mathcal{H}_P = \mathcal{H}_{P'} \otimes \mathcal{H}_T \oplus \mathcal{H}_Q$ such that $R_i(|\psi\rangle_{P'} \otimes |j\rangle_S) = |\psi\rangle_{P'} \otimes |\phi_{ij}\rangle_T$ and $E_i|\psi\rangle_{P'}|j\rangle_S = |\psi\rangle_{P'}|\varphi_{ij}\rangle_S$. The desired subspace is given by $\mathcal{H}_{P'} \otimes |0\rangle_T$ for any base state $|0\rangle_T$ of T . Note that the desired isometry is implicitly defined via the two subsystem decompositions. ■

The maps F_{ij} defined in the statement of Lemma 9 are well defined from \mathcal{V} to \mathcal{H}_I . Let M, N be the dimensions of \mathcal{V} and \mathcal{H}_I , respectively. By choosing orthonormal bases $\{|i\rangle_{\mathcal{V}}\}_i$ of \mathcal{V} and $\{|j\rangle_{\mathcal{H}_I}\}_j$ of \mathcal{H}_I , the F_{ij} are expressible as $N \times M$ matrices (also denoted by F_{ij}) with entries $(F_{ij})_{kl} = {}^I\langle k|F_{ij}|l\rangle_{\mathcal{V}}$. Without loss of generality, $M \geq N$, for otherwise the subsystem I' is clearly not protectable. The condition in Lemma 9 can be seen to be equivalent to the requirement that there exists a unitary matrix V such that $F_{ij}V$ contains a multiple of the $N \times N$ identity matrix as its first $N \times N$ block. The code \mathcal{D} is spanned by the first N columns of V . This requirement is reminiscent of the familiar condition on the existence of an N -dimensional error-detecting quantum code, according to which there must exist a unitary matrix W such that WE_iW^\dagger has a multiple of an $N \times N$ identity matrix as its first diagonal subblock. The protectability requirement can indeed be reduced to the existence of an error-detecting code. In particular, I' is protectable if and only if there exists an N -dimensional error-detecting code for the operators $\{F_{ij}^\dagger F_{kl}\}$. Note that this is equivalent to requiring the existence of an N -dimensional error-correcting code for the operators F'_{ij} , where F'_{ij} is the square matrix obtained from F_{ij} by expanding with rows of zeros. However, we do not have to consider all operators $F_{ij}^\dagger F_{kl}$. It suffices to find an N -dimensional error-detecting code for operators of the form $F_{ij}^\dagger F_{ij}$ and $F_{ij}^\dagger F_{\pi(ij)}$, where π is a cyclic permutation of the index pairs.

We call subspaces \mathcal{D} satisfying the condition in Lemma 9 “protecting” codes (for I'). There are several procedures that can be used to reduce the difficulty of the search for protecting codes.

Lemma 10 *All protecting codes are contained in the null space of the F in the linear span of $\{F_{ij}\}_{ij}$ whose rank is strictly less than N .*

Proof. Let V be as specified in the paragraph before the statement of the lemma. If the rank of F is less than N , then the first $N \times N$ block of FV must be zero, from which the result follows. ■

Let G_1, \dots, G_k be $N \times M$ matrices. We say that $\{G_1, \dots, G_k\}$ has maximal row rank if the span of the rows of the G_i has dimension kN . The next lemma generalizes Lemma 10.

Lemma 11 *Let G_1, \dots, G_k be in the span of the F_{ij} such that $\{G_1, \dots, G_k\}$ does not have maximal row rank, but for every $k-1$ independent G'_1, \dots, G'_{k-1} in the span of the G_i , $\{G'_1, \dots, G'_{k-1}\}$ has maximal row rank. Then any protecting codes are contained in the intersection of the null spaces of the G_i .*

Proof. Let V be such that $G_i V$ has an initial block proportional to the $N \times N$ identity matrix and \mathcal{D} is spanned by the first N columns of V . Suppose that \mathcal{D} is not contained in the null space of some G_i . Then $G_i V$'s initial $N \times N$ block is not zero. The space \mathcal{G} of matrices G in the span of the G_j such that GV has an initial $N \times N$ zero block is $k-1$ -dimensional. Because the row

span of G_i is independent of the linear span \mathcal{R} of the rows of the matrices in \mathcal{G} , the dimension of \mathcal{R} is strictly less than $(k-1)N$, contradicting the assumption of the lemma. ■

Lemma 11 means that in principle, the problem of finding \mathcal{D} can be reduced to the case where each F_{ij} has full rank and its row space is independent of the space spanned by the rows of the other F_{kl} . In this case there are at most M/N independent matrices F_{ij} . Unfortunately, we do not know of an efficient algorithm for checking the condition of Lemma 11 that would enable reducing the problem to this case. Nevertheless, we can show that one can reduce to the case where there are at most $M-1$ independent F_{ij} .

Lemma 12 *For $N > 1$, if there are M or more independent F_{ij} , then there exists a nonzero G in the span of the F_{ij} such that G does not have full rank.*

Proof. Let $\{G_i\}_{i=1}^l$ be a basis of the linear span of the F_{ij} . Let g_i^j be the j 'th row of G_i . If one of the g_i^j is zero, we are done. Suppose $l > M$. Then the g_i^1 are dependent, so there is a non-trivial linear combination of the G_i with zero first row. Suppose $l = M$. Consider the matrices A_j whose i 'th rows are the g_i^j . Then there exists a non-zero linear combination $\alpha A_1 + \beta A_2$ with determinant zero. Let $x \neq 0$ be in the null space of $(\alpha A_1 + \beta A_2)^T$. Then $G = \sum_i x_i G_i$ is not zero and the row vector $y = (\alpha, \beta, 0, \dots)$ satisfies $yG = 0$, so that G does not have full rank. ■

Note that the proof of the lemma contains an efficient algorithm for finding a non-full rank G .

Let $\rho^{(AB)}$ be a density matrix on systems A and B . What states σ of A can be obtained by projecting B onto a pure state $|\psi\rangle_B$? The “pure σ -projection problem” for $\rho^{(AB)}$ is to determine a state $|\psi\rangle_B$ such that ${}^B\langle\psi|\rho^{(AB)}|\psi\rangle_B = p\sigma$ for some $p \neq 0$, if such a state exists.

Theorem 13 *If the span of the rows of the F_{ij} is M -dimensional, the problem of determining whether subsystem l' is protectable is efficiently reducible to a pure \mathbf{I} -projection problem.*

If the rows of the F_{ij} do not span the full space, then the protectability problem may be reduced to a generalization of the pure \mathbf{I} -projection problem. However, in situations where the original error operators are associated with quantum operations, the F_{ij} 's do not have a common null space, even after the restrictions of the previous lemmas have been applied. Otherwise there would be states for which all E_i have zero probability.

Proof. Let G_1, \dots, G_k be a basis for the linear space spanned by the F_{ij} . We can choose an orthonormal basis of \mathcal{V} such that in this basis, the matrices G_i have a block form $[G_{i1}, G_{i2}, \dots, G_{ii}, 0, \dots, 0]$, where the G_{ij} are $N \times N_j$ matrices of full rank. We attempt to find the desired subspace \mathcal{D} by choosing an orthonormal basis for \mathcal{D} . Let X be the matrix whose columns are members of this basis. We wish to solve the k identities $\alpha_i \mathbf{I} = G_i X$ for X and $\alpha = (\alpha_i)_i$. We can write X in block form, $X = [X_1; \dots; X_k]$, where X_i is $N_j \times N$ and the X_i are placed one above the other. The desired identities can be expanded as

$$\alpha_i \mathbf{I} = \sum_{j=1}^i G_{ij} X_j. \quad (14)$$

The X_j can be eliminated by solving the equations in order. That is, from $\alpha_1 \mathbf{I} = G_{11} X_1$ we obtain $\alpha_1 = 0$ and $X_1 = 0$ if $N_1 \neq N$, and $X_1 = \alpha_1 G_{11}^{-1}$ otherwise. We write this as a linear constraint $L_1 \cdot \alpha = 0$ and an identity $X_1 = \alpha_1 \tilde{G}_{11}$, where L_1 may be “empty” (if $N_1 = N$) and we set \tilde{G}_{11}

to be any left inverse of G_{11} . Once we have obtained $X_j = \sum_m \alpha_m \tilde{G}_{jm}$ and linear constraints $L_j \alpha = 0$ for $j < i$, we can solve for X_i by substituting in Eq. 14:

$$G_{ii}X_i = \alpha_i \mathbf{I} - \sum_{j=1}^{i-1} \sum_{m=1}^j \alpha_m G_{ij} \tilde{G}_{jm}. \quad (15)$$

The right hand side of this identity is a matrix H_i that depends linearly on α . $G_{ii}X_i = H_i$ can be solved if and only if the columns of H_i are in the column span of G_{ii} . This condition yields a set of linear constraints $L_i \alpha = 0$. If the constraints are satisfied, then we can compute $X_i = G'_{ii} H_i$, where G'_{ii} is a left inverse of G_{ii} . We can therefore define \tilde{G}_{im} by the identity $X_i = \sum_{m=1}^i \alpha_j \tilde{G}_{im}$. At the end of this process, the only free variables remaining are the α_j , which must be chosen to satisfy the orthonormality constraint on X , $\sum_i X_i^\dagger X_i = \mathbf{I}$. Expanding, we get

$$\sum_i \sum_{jk} \bar{\alpha}_j \alpha_k \tilde{G}_{ij}^\dagger \tilde{G}_{ik} = \mathbf{I}, \quad (16)$$

subject to $L_i \alpha = 0$ for all i . If the linear constraints cannot be solved, we are done. Define $\rho^{(\text{AB})}$ by

$$\rho^{(\text{AB})} = t \sum_i \sum_{jk} \tilde{G}_{ij}^{(\text{A})\dagger} \tilde{G}_{ik}^{(\text{A})} |j\rangle_{\text{B}} \langle k|, \quad (17)$$

where t is chosen so that $\text{tr}(\rho^{(\text{AB})}) = 1$. Any state $|\psi\rangle_{\text{B}}$ in the subspace defined by $L_i |\psi\rangle_{\text{B}} = 0$ (with L_i defined with respect to the basis consisting of the $|j\rangle_{\text{B}}$) that solves the pure I-projection problem yields a solution for α by letting α_j be a suitably scaled multiple of the coefficient of $|j\rangle_{\text{B}}$ in the solution. It follows that to complete the proof, it suffices to project $\rho^{(\text{AB})}$ onto the subspace of B satisfying the linear constraints L_i and renormalize the resulting positive semidefinite operator. This operator is a density matrix for which the pure I-projection problem is equivalent to the problem of whether l' is protectable. ■

The pure I-projection problem may be reduced to a problem of finding special matrices in a linear space of matrices.

Theorem 14 *The pure I-projection problem is polynomially equivalent to the problem of finding a matrix with orthonormal columns in a linear space of matrices.*

Proof. Consider the pure I-projection problem for $\rho^{(\text{AB})}$. By purifying $\rho^{(\text{AB})}$ with the addition of an environment E , we obtain a pure state $|\psi\rangle_{\text{ABE}}$ whose reduced density matrix on AB is $\rho^{(\text{AB})}$. The pure I-projection problem is now equivalent to the problem of finding $|\phi\rangle_{\text{B}}$ such that ${}^{\text{B}}\langle\phi||\psi\rangle_{\text{ABE}}$ is proportional to a maximally entangled state between A and E. Note that without loss of generality, the dimension of E is greater than that of A. Otherwise, the problem has no solution. We can expand everything in a basis for the different systems' Hilbert spaces: $|\phi\rangle_{\text{B}} = \sum_i \alpha_i |i\rangle_{\text{B}}$, $|\psi\rangle_{\text{ABE}} = \sum_{ijk} m_{ijk} |i\rangle_{\text{A}} |j\rangle_{\text{B}} |k\rangle_{\text{E}}$. Let M_j be the matrix with coefficients $(M_j)_{ki} = m_{ijk}$. The property that ${}^{\text{B}}\langle\phi||\psi\rangle_{\text{ABE}}$ is maximally entangled is equivalent to the property that $\sum_j \alpha_j M_j$ has orthonormal columns.

Given any set of matrices M'_j we can reverse the reduction of the previous paragraph by setting $M_j = t M'_j$ with $t = 1/(\sum_j \text{tr}(M'^{\dagger}_j M'_j))$ to obtain a state such that its pure I-projection problem is equivalent to the problem of finding a matrix with orthonormal columns in the span of the M'_j . ■

Whether there is an efficient algorithm for finding a matrix with orthonormal columns in a linear space of matrices is an open question.

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- [1] E. Knill, R. Laflamme, and L. Viola, Phys. Rev. Lett. **84**, 2525 (2000).
 - [2] D. Kribs, R. Laflamme, and D. Poulin, Phys. Rev. Lett. **94**, 180501/1 (2005).
 - [3] L. Viola, E. Knill, and R. Laflamme, J. Phys. A **34**, 7067 (2001).
 - [4] M.-D. Choi and D. W. Kribs (2005), quant-ph/0507213.
 - [5] D. W. Kribs, R. Laflamme, D. Poulin, and M. Lesosky (2005), quant-ph/0504189.
 - [6] M. A. Nielsen and D. Poulin (2005), quant-ph/0506069.
 - [7] P. Zanardi, Phys. Rev. A **63**, 012301/1 (1999).
 - [8] R. Alicki (2004), quant-ph/0411008.
 - [9] P. Aliferis, D. Gottesman, and J. Preskill (2005), quant-ph/0504218.
 - [10] T. C. Ralph, W. J. Munro, and G. J. Milburn (2001), quant-ph/0110115.
 - [11] T. C. Ralph, A. Gilchrist, G. J. Milburn, W. J. Munro, and S. Glancy, Phys. Rev. A **68**, 042319/1 (2003).
 - [12] A. Shabani and D. A. Lidar (2005), quant-ph/0505051.
 - [13] Strictly speaking these are “quantum” subsystems. One can also consider classical subsystems. Classical subsystems require an explicit basis for encoding classical information. See, for example, [1].
 - [14] J. Kempe, D. Bacon, D. A. Lidar, and K. B. Whaley, Phys. Rev. A **63**, 042307/1 (2001).
 - [15] D. Poulin (2005), quant-ph/0508131.
 - [16] D. Bacon (2005), quant-ph/0506023.
 - [17] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
 - [18] D. G. Cory, A. F. Fahmy, and T. F. Havel, in *Proceedings of the 4th Workshop on Physics and Computation*, edited by T. T. *et al.* (New England Complex Systems Institute, Boston, Massachusetts, 1996), pp. 87–91.
 - [19] N. A. Gershenfeld and I. L. Chuang, Science **275**, 350 (1997).
 - [20] E. Knill and R. Laflamme, Phys. Rev. Lett. **81**, 5672 (1998).
 - [21] E. Knill and R. Laflamme, Phys. Rev. A **55**, 900 (1997).
 - [22] T. W. Hungerford, *Algebra* (Springer Verlag, New York, 1980).
 - [23] C. A. Struble, Ph.D. thesis, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, US (2000), available online at <http://scholar.lib.vt.edu/theses/available/etd-04282000-13520019>.